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Note on the Riccati Method for Differential Eigenvalue Problems of Odd Order

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A technique is described for traversing singular points during the numerical evaluation of eigenvalues of a system of linear ordinary differential equations using the Riccati method. The technique may be applied to a system of even or odd order and with any distribution of linear homogeneous boundary conditions. Comparison is made with a method which uses a complex contour of integration to avoid the singularities.

1. INTRODUCTION

This note is concerned with the Riccati transformation method for the computation of eigenvalues of a system of linear ordinary differential equations of the form

$$\frac{d\mathbf{y}}{dz} = L(z, \sigma)\,\mathbf{y} \tag{1}$$

subject to the linear separated boundary conditions

$$B\mathbf{y}(0) = \mathbf{0}, \tag{2a}$$

$$C\mathbf{y}(x) = \mathbf{0}. \tag{2b}$$

Here $\mathbf{y}$ is a real n-vector and L is an $n \times n$ matrix which depends on the independent variable z and on some scalar eigenparameter σ. The real matrices B and C have full rank and their dimensions are $k \times n$ and $l \times n$, respectively, where $k + l = n$ and $k \geqslant l$. If (1) is solved by a traditional shooting method which operates by generating a basis of the solution space, then difficulties are encountered if the real parts of the eigenvalues of L are widely separated. Steps have to be taken to overcome the effects of growth in the basis components. The Riccati method overcomes some of these growth problems. Scott [1, 2] first described the use of the Riccati method for the problem defined by (1) and (2): he considered the case in which $k = l = m$ and $n = 2m$, with $B = [I\ 0]$ and $C = [I\ 0]$ or $[0\ I]$, where I is the unit $m \times m$ matrix. Sloan and Wilks [3] considered this even-order problem for general matrices B and C of dimensions $m \times 2m$ and rank m. In a recent paper Davey [4] has described the use of the Riccati method for system (1) when the order is even or odd.

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The Riccati method of solution involves the integration of a system of nonlinear Riccati differential equations along the real line segment from $z = 0$ to $z = x$. In the course of this integration it is usually necessary to traverse points at which the dependent variables become singular. It is shown in [2, 3] that in the even-order case, $n = 2m$, the singular points may be avoided by a procedure which involves the inversion of an $m \times m$ matrix and a switch to a new set of dependent variables. Denman [5] first introduced the interesting idea of using a complex contour of integration as a means of traversing singularities and Davey [4] used this technique for the solution of an odd-order problem. The contour integration method is applicable to even- or odd-order systems with $k + l = n$ and $k \geqslant l$. In this note we show that the switching procedure which is used in [2, 3] for the even-order case with $k = l$ may be extended to deal with even- or odd-order systems with $k \geqslant l$. Davey illustrated the complex contour method by considering the evaluation of eigenvalues arising out of perturbations of the Blasius profile, this being a problem on a semi-infinite interval with $n = 3$ and $k = 2$. Here we consider the same illustrative problem and it is shown that the extended switching procedure has certain advantages over the complex contour method.

2. Switching Procedure

Introduce vectors $\mathbf{U}(z)$ and $\mathbf{V}(z)$ with k and l components, respectively, using the transformation

$$\mathbf{U}(z) = B\mathbf{y}(z), \qquad \mathbf{V}(z) = D\mathbf{y}(z), \tag{3}$$

where the constant $l \times n$ matrix D is chosen such that $M = \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is nonsingular. System (1) may be written in terms of the n-vector $\mathbf{Y}(z) = \left[\begin{smallmatrix} \mathbf{U}(z) \\ \mathbf{V}(z) \end{smallmatrix}\right]$ and the transformed system may be partitioned in the form

$$\begin{aligned} \frac{d\mathbf{U}}{dz} &= \mathscr{A}(z, \sigma)\,\mathbf{U} + \mathscr{B}(z, \sigma)\,\mathbf{V}, \\ -\frac{d\mathbf{V}}{dz} &= \mathscr{C}(z, \sigma)\,\mathbf{U} + \mathscr{D}(z, \sigma)\,\mathbf{V}, \end{aligned} \tag{4}$$

where the matrices $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ have dimensions $k \times k$, $k \times l$, $l \times k$, $l \times l$, respectively. After transformation, the boundary conditions (2) take the form

$$\mathbf{U}(0) = \mathbf{0}, \tag{5a}$$

$$\alpha\mathbf{U}(x) + \beta\mathbf{V}(x) = \mathbf{0}, \tag{5b}$$

where $[\alpha\ \beta] = CM^{-1}$. The Riccati method involves the introduction of a $k \times l$ matrix $E(z)$ by means of the transformation

$$\mathbf{U}(z) = E(z)\,\mathbf{V}(z). \tag{6}$$

If $\Sigma(z)$ denotes the space of solutions $\mathbf{Y}(z) = [{\mathbf{U}(z) \atop \mathbf{V}(z)}]$ of (4) which satisfy the initial condition (5a), then at any station z this space will be a vector space of dimension l. The use of the transformation (6) assumes that any $\mathbf{Y}(z) \in \Sigma(z)$ may be represented as a linear combination of the columns of a matrix

$$\begin{bmatrix} E(z) \\ I \end{bmatrix} V(z), \tag{7}$$

where I is the unit $l \times l$ matrix, and the columns of the $l \times l$ matrix $V(z)$ are linearly independent and they may be regarded as a basis for the solution $\mathbf{V}(z)$. With $\mathbf{Y}(z)$ represented by (7), $\mathbf{U}(z)$ and $\mathbf{V}(z)$ will be solutions of (4) if the $k \times l$ matrix $E(z)$ satisfies the Riccati equation

$$E' = \mathscr{B} + \mathscr{A}E + E\mathscr{D} + E\mathscr{C}E, \tag{8}$$

where the prime denotes d/dz. The boundary conditions to be imposed on $E(z)$ have been discussed by Sloan [6] for the case $k = l$ and these arguments apply equally well to the case $k \neq l$. The initial condition (5a) is satisfied by a linear combination of the basis elements (7) if and only if $E(0) = 0$. If we consider the space $\Sigma(z)$ for $z > 0$, we see that the terminating condition (5b), or $[\alpha\ \beta]\,\mathbf{Y}(x) = \mathbf{0}$, will be satisfied at any point $z = x$ where there is a vector $\mathbf{Y}(z)$ in $\Sigma(z)$ and in $N([\alpha\ \beta])$, where $N([\cdot])$ denotes the null space of $[\cdot]$. With $\mathbf{Y}(z)$ represented by the basis (7) a necessary and sufficient condition for the existence of such a common vector is that

$$\det[\alpha E(x) + \beta] = 0. \tag{9}$$

For prescribed x, eigenvalues of the problem defined by Eqs. (1) and (2) are those values of the parameter σ for which (9) is satisfied, where $E(x)$ is obtained by integrating (8) over the range $0 \leqslant z \leqslant x$ from an initial state $E(0) = 0$.

For the case $k = l$, Sloan [6] has pointed out that $\det[E(z)]$ will be singular at any point z where $\Sigma(z)$ and the null space of the $l \times n$ matrix $[0\ I]$ have a vector $\mathbf{Y}(z)$ in common. For any integers k and l satisfying $k + l = n$, the columns of (7) cannot be used as a basis at any point $z = z_0$ where there is a vector $\mathbf{Y}(z) \in \Sigma(z) \cap N([0\ I])$, with 0 and I denoting the zero $l \times k$ and the unit $l \times l$ matrices, respectively. Near $z = z_0$ the structure (7) does not provide a proper representation of the solution space and if the integration of (8) approaches $z = z_0$, then elements of $E(z)$ will become unbounded. In terms of the original dependent variable $\mathbf{y}(z)$, a singularity in $E(z)$ will occur at any point where there is a vector $\mathbf{y}(z) \in \Sigma(z) \cap N(D)$, and it follows that the choice of D will affect the positions of singular points.

If the integration of (8) approaches a singular point, remedial steps have to be taken and Scott [2] has explained that, in the case $k = l$, the singularity may be traversed by switching to the inverse matrix $E^{-1}(z)$. Davey [4] has shown that for any k and l with $k + l = n$, singularities may be avoided by deforming the contour of integration into the complex plane. Here we propose an extension of the linear transformation used in [3] as a means of traversing the singularity in the general case with $k + l = n$. Introduce new dependent vectors $\phi(z)$ and $\eta(z)$ through the linear transformation

$$\begin{bmatrix} \mathbf{U}(z) \\ \mathbf{V}(z) \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{bmatrix} \begin{bmatrix} \phi(z) \\ \eta(z) \end{bmatrix} = J \begin{bmatrix} \phi(z) \\ \eta(z) \end{bmatrix} \tag{10}$$

where J is a constant nonsingular matrix and the submatrices ϵ_1, ϵ_2, ϵ_3, and ϵ_4 have dimensions $k \times k$, $k \times l$, $l \times k$, and $l \times l$, respectively. If $\phi(z)$ and $\eta(z)$ are related by

$$\phi(z) = G(z)\, \eta(z), \tag{11}$$

then it is readily shown that at $z = \bar{z}$,

$$G(\bar{z}) = (\epsilon_1 - E(\bar{z})\, \epsilon_3)^{-1}(E(\bar{z})\, \epsilon_4 - \epsilon_2), \tag{12}$$

provided $(\epsilon_1 - E(\bar{z})\, \epsilon_3)$ is nonsingular. In terms of $\mathbf{\Phi}(z) = \left[\begin{smallmatrix} \phi(z) \\ \eta(z) \end{smallmatrix}\right]$ the given system (1) takes the form $\mathbf{\Phi}' = \mathscr{L}(z, \sigma)\, \mathbf{\Phi}$, where $\mathscr{L} = J^{-1}ML(J^{-1}M)^{-1}$. This may be partitioned as in (4) and the Riccati equation in $G(z)$, analogous to Eq. (8), is readily obtained. If in the course of integrating Eq. (8), a point $\bar{z}$ is reached where some norm of $E(z)$ exceeds a preset value, a switch is made to $G(z)$ via (12) and the singular point is traversed using the Riccati system in $G(z)$. If desired, a return may be made to $E(z)$ beyond the singularity. If $G(z)$ remains well behaved, the integration may be continued in the $G(z)$ system as far as $z = x$. In terms of $\phi(z)$ and $\eta(z)$ the terminating condition (5b) takes the form $\gamma\phi(x) + \delta\eta(x) = \mathbf{0}$, where $\gamma = \alpha\epsilon_1 + \beta\epsilon_3$ and $\delta = \alpha\epsilon_2 + \beta\epsilon_4$. If J is chosen so that $\gamma \neq 0$, the eigenvalues may be obtained using the terminating condition

$$\det[\gamma G(x) + \delta] = 0. \tag{13}$$

There is obviously a great deal of flexibility in the choice of the matrix J in transformation (10). If a transformation at $z = \bar{z}$ is such that for $z \in [\bar{z}, x]$ the solution space $\Sigma(z)$ contains no vector $\mathbf{\Phi}(z) = \left[\begin{smallmatrix} \phi(z) \\ \eta(z) \end{smallmatrix}\right]$ with $\eta(z) = \mathbf{0}$, then the $G(z)$ system may be integrated from $z = \bar{z}$ to $z = x$. One of the transformations described in the next section possesses these rather fortunate properties.

If the inverse of matrix J in Eq. (10) is $\left[\begin{smallmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{smallmatrix}\right]$, with appropriate partitioning, then $G(z)$ will have singular elements at any z where there is a vector $\mathbf{Y}(z) \in \Sigma(z) \cap N([\kappa_3\ \ \kappa_4])$. It seems appropriate, therefore, to select the transformation in such a way that at $z = \bar{z}$ any linear combination of columns of (7) be orthogonal to $N([\kappa_3\ \ \kappa_4])$. This may be achieved if $[\kappa_3\ \ \kappa_4] \equiv P[E(\bar{z})^T\, I]$, where P is any nonsingular $l \times l$ matrix. Subject to this constraint, the simplest choice for J^{-1} is now $\left[\begin{smallmatrix} I & 0 \\ \kappa_3 & \kappa_4 \end{smallmatrix}\right]$ with κ_3 and κ_4 as above, and transformation (12) may now be written as

$$G(\bar{z}) = (I + E(\bar{z})\, E(\bar{z})^T)^{-1}\, E(\bar{z})\, P^{-1}. \tag{14}$$

If required, P may be used to scale the elements of $G(\bar{z})$. Note that the implementation of (14) involves the inversion of a symmetric matrix which has eigenvalues bounded below by unity. If the elements of $E(z)$ are monitored during the integration of (8) and $\bar{z}$ is selected such that $\| \mathbf{s} \|_2 < c$ at $z = \bar{z}$, where c is a preset constant, $\mathbf{s}$ is any column of $E(z)^T$ and $\| \cdot \|_2$ is the Euclidean vector norm, then each element of $E(\bar{z})\, E(\bar{z})^T$ will be less than c^2 in modulus. This imposes the controllable limit $1 + kc^2$ on the condition number of $I + E(\bar{z})\, E(\bar{z})^T$ in terms of the spectral matrix norm. In this symmetric case the condition number is the ratio of the largest eigenvalue to the smallest eigenvalue. A limited condition number should prevent the introduction of large rounding errors during the switching operation.

3. Illustrative Example

The differential equation associated with the Blasius velocity profile perturbation problem is [4]

$$y''' + fy'' + \sigma f'y' + (1 - \sigma) f''y = 0 \tag{15}$$

and the boundary conditions are

$$y = y' = 0 \qquad \text{at} \quad z = 0, \tag{16}$$

$$y' \to 0 \text{ exponentially as } z \to \infty, \tag{17}$$

where f is the Blasius solution. Equation (15) may be written in the format of (4) with $\mathbf{U} = [^{y}_{y'}]$ and $\mathbf{V} = [y'']$. From this system we obtain the components of the Riccati equation (8) as

$$\begin{aligned} E_1' &= E_2 + fE_1 + (1 - \sigma) f'' E_1^2 + \sigma f' E_1 E_2 , \\ E_2' &= 1 + fE_2 + (1 - \sigma) f'' E_1 E_2 + \sigma f' E_2^2, \end{aligned} \tag{18}$$

where $E = [^{E_1}_{E_2}]$. If Eqs. (18) are integrated from $z = 0$ with initial state $E(0) = 0$, then, as pointed out in [4], a singularity is encountered. Two switching procedures were considered each of which permitted integration to large z.

Procedure 1

Let

$$\phi = \begin{bmatrix} y' \\ y'' \end{bmatrix}, \qquad \eta = [y], \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{19}$$

and obtain Riccati equations

$$\begin{aligned} G_1' &= G_2 - G_1^2, \\ G_2' &= -(1 - \sigma) f'' - \sigma f' G_1 - fG_2 - G_1 G_2 . \end{aligned} \tag{20}$$

Transformation (12) has the simple form

$$G_1(\bar{z}) = E_2(\bar{z})/E_1(\bar{z}), \qquad G_2(\bar{z}) = 1/E_1(\bar{z}). \tag{21}$$

With this procedure a switch was made from $E(z)$ to $G(z)$ at the first monitoring point where $\| E \|_2$ exceeded unity, where E is here regarded as a two-vector, and $\| \cdot \|_2$ again denotes the Euclidean vector norm.

Procedure 2

The second procedure used transformation (14) with the scalar P^{-1} set to $(1 + \| E(\bar{z})\|_2^2)/\| E(\bar{z})\|_\infty$, where $\| \cdot \|_\infty$ denotes the maximum vector norm which is the modulus of the largest element in the vector. As in the first approach a switch was made at the first monitoring point $z = \bar{z}$ where $\| E(z)\|_2$ exceeded unity. If $E_1(\bar{z}) = e_1$ and $E_2(\bar{z}) = e_2$, the Riccati equations in $G(z)$ have the form

$$\begin{aligned} G_1' &= G_2 + (f - e_2)\, G_1 + c_1 G_1^2 + c_2 G_1 G_2, \\ G_2' &= P^{-1} - e_1 G_1 - e_2 G_2 + (f - e_2)\, G_2 + c_1 G_1 G_2 + c_2 G_2^2, \end{aligned} \tag{22}$$

where $c_1 = P[e_1 e_2 + (1 - \sigma) f'' - e_1 f]$, $c_2 = P[e_2^2 - e_1 + \sigma f' - e_2 f]$. Transformation (14) has the form

$$G_1(\bar{z}) = E_1(\bar{z})/\| E(\bar{z})\|_\infty, \qquad G_2(\bar{z}) = E_2(\bar{z})/\| E(\bar{z})\|_\infty. \tag{23}$$

With each procedure the G system remained bounded between $z = \bar{z}$ and the point at which the terminating boundary condition was imposed.

Wilks and Bramley [7] have discussed the asymptotic behavior of the solutions. Their boundary conditions for large z required for the isolation of exponentially decaying $y'(z)$ may be written as

$$\alpha \mathbf{U}(z) + \beta \mathbf{V}(z) \to \mathbf{0} \qquad \text{as} \quad z \to \infty, \tag{24}$$

where $\alpha = [0 \;\; h(z)]$ and $\beta = 1$, with $h(z) = (z - q_1)\{1 + (1 - \sigma)(z - q_1)^{-2}\}$ and $q_1 = 1.21676$. Condition (24) enables us to obtain terminating conditions on $G(z)$ analogous to Eq. (13). For procedures 1 and 2 the conditions are, respectively,

$$h(z)\, G_1(z) + G_2(z) \to 0, \tag{25}$$

and

$$1 - P\{e_1 G_1(z) + (e_2 - h(z))\, G_2(z)\} \to 0 \tag{26}$$

as $z \to \infty$. The aim is to integrate the G systems to a large value of z, say $z = x_\infty$, and to obtain the eigenvalues iteratively by finding values of σ for which the left-hand sides of (25) and (26) are zero.

Before discussing possible methods of obtaining initial estimates for x_∞ it is essential to describe a technicality concerning the use of the above procedures. Procedure 1 was found to behave extremely well over a wide range of eigenvalues, whereas procedure 2 became unreliable for higher eigenvalues. In both cases the switch from the E system to the G sytem was performed without loss of accuracy. The source of error was obtained by a consideration of the nature of the elements of the matrix $G(z)$ at a typical station $z = x$. Equation (11) indicates that $\phi(x) = G(x)\,\eta(x)$ and it follows that if $\eta(x) = \mathbf{e}_i$, the ith column of the unit $l \times l$ matrix, then the ith column of $G(x)$ may be identified with $\phi(x)$. Hence the ith column of $G(x)$ is equal to that $\phi(x)$ which is obtained by solving the given system of differential equations on the interval $0 \leqslant z \leqslant x$, with the prescribed homogeneous boundary conditions at $z = 0$ and with the inhomogeneous condition $\eta(x) = \mathbf{e}_i$ at $z = x$. This view of the Riccati elements is analogous to that adopted in the invariant imbedding approach as described, for example, in the text by Scott [1].

If the ideas outlined above are applied to the Riccati system in procedure 1, it is readily seen that at any station $z = x$, $G_1(x) = y'(x)$ and $G_2(x) = y''(x)$, where $y(z)$ is the solution of (15) and (16) in $0 \leqslant z \leqslant x$ with the additional constraint $y(x) = 1$. For the eigenvalue problem under discussion $y'(z)$ and $y''(z)$ decrease exponentially as $z \to \infty$ so the G system in procedure 1 might well be described as the natural choice for this problem. For procedure 2 it is readily shown that $G_1(x) = y(x)$ and $G_2(x) = y'(x)$, where $y(z)$ is the solution of (15) and (16) in $0 \leqslant z \leqslant x$ with the additional constraint $P(e_1\, y(x) + e_2\, y'(x) + y''(x)) = 1$. The terms $y'(x)$ and $y''(x)$ decrease to zero as $x \to \infty$ and it follows that $G_1(z)$ tends to the limit $1/Pe_1$ as $z \to \infty$. This particular limit was found to be the source of the error in the use of procedure 2 for higher eigenvalues. Roundoff errors produced by differencing were introduced during the evaluation of the expression $P^{-1} - e_1 G_1$ on the right-hand side of the second equation in (22). To circumvent this difficulty $G_1(z)$ was replaced by $F_1(z) = G_1(z) - 1/Pe_1$ and the integration was effected in terms of $F_1(z)$ and $G_2(z)$ with the terminating condition (26) replaced by

$$h(z)\, G_2(z) - e_1 F_1(z) - e_2 G_2(z) \to 0. \tag{27}$$

The modified procedure 2 gave accurate results over a wide range of eigenvalues.

One of the main problems in the numerical solution of a system of ordinary differential equations defined on an infinite interval is the determination of the point x_∞ at which the terminating boundary conditions are applied. This problem has been considered for a second-order inhomogeneous system in interesting and useful papers by Robertson [8] and Alspaugh [9]. Robertson used a matrix factorization method and Alspaugh employed invariant imbedding, the approaches being related in that each involves a double sweep with criteria imposed at the end of the forward sweep to determine x_∞. For the eigenvalue problem under consideration which differs from the inhomogeneous problem in that it necessarily involves an iteration with respect to the eigenparameter σ, an initial estimate of x_∞ was obtained from an examination of the behavior of the Riccati elements as z was increased.

The element $G_2(z)$ defined in procedure 1 oscillates with respect to z as z increases.

If $\sigma = \sigma_i^-$, where σ_i denotes the ith positive eigenvalue and σ_i^- denotes a value slightly less than σ_i, the element has $i - 1$ zeros and $G_2(z) \to 0$ from above or below as $z \to \infty$ according to whether i is even or odd. If $\sigma = \sigma_i^+$ the element $G_2(z)$ has i zeros and the location of the ith zero tends to ∞ as $\sigma \to \sigma_i$ from above, whereas the locations of the first $i - 1$ zeros are insensitive to changes in σ for σ near σ_i. When using procedure 1 for the evaluation of σ_i an approximation to σ_i was selected and the integration was carried out to a point x_∞ beyond the $(i - 1)$th zero of $G_2(z)$. The eigenvalue for the problem defined over this finite interval was obtained iteratively and the whole process was repeated for progressively increasing x_∞ until further increase had no effect on the computed value of σ. For $\sigma = \sigma_i^-$ the element $G_2(z)$ in the modified procedure 2 has $i - 3$ zeros for $i > 3$ and $i - 1$ zeros for $i \leqslant 3$. In all cases $G_2(z) \to 0$ from above or below as $z \to \infty$ according to whether i is odd or even. $G_2(z)$ has an additional zero if σ is increased to σ_i^+ and the location of this zero tends to infinity as $\sigma \to \sigma_i$ from above. When using this procedure the initial estimate of x_∞ was made so that the finite interval $[0, x_\infty]$ contained the first set of zeros of $G_2(z)$ and the iteration was then performed as described above.

4. Results and Comments

Several eigenvalues were obtained using procedure 1 and the modified procedure 2 as described above and the results agreed with those given in [7]. For example, the values 2.0000, 5.6287, and 19.0397 were obtained for σ_1, σ_3, and σ_{10} with final x_∞ values of 6.0, 7.5, and 14, respectively. At $\sigma = \sigma_i$ the approximate locations of the highest zeros of $G_2(z)$ as defined in procedure 1 are 4.2 and 8.6 for $i = 3$ and 10, respectively. For $G_2(z)$ as defined in procedure 2 the highest zeros for $i = 3$ and 10 have approximate locations 3.7 and 8.4, respectively. The equations were integrated using a standard fourth-order variable step Runge-Kutta procedure with stepsize control based on local error. Computations were performed on an ICL 1904S computer using single length arithmetic. The switching methods described in Section 3 proved to be effective in traversing the singularity in this odd-order Riccati formulation. It was suggested in the discussion at the end of Section 3 that procedure 1 might be described as the natural formulation for this problem: this remark derived from a consideration of the domain of the problem and not from difficulties encountered at the switching stage. The Riccati formulation is not unique and any relevant information which is available about a particular problem might well be utilized in selecting a Riccati formulation.

The eigenvalues σ_1, σ_3, and σ_{10} were also obtained by integrating over the complex contour $z = t - 0.02it(x_\infty - t)$, $0 \leqslant t \leqslant x_\infty$, as described by Davey [4]. In this complex formulation the number of real differential equations is doubled and one might expect a consequent decrease in efficiency. The computer time used by the complex contour method was found to be greater than that used by the methods described in Section 3 by a factor which was approximately 3 for the eigenvalue σ_1

and 4 for the eigenvalue σ_3. Convergence difficulties were encountered with the contour method for the eigenvalue σ_{10}.

One technical advantage of the switching method over the contour method is that with the former the eigenfunction is readily computed. The method described by Sloan [6] applies to the case of uneven boundary conditions if the switching techniques of Section 3 are employed. The eigenfunction $y_i(z)$ associated with eigenvalue σ_i, $i = 1, 3, 10$—normalized so that $y''(0) = 1$—was computed using this method. The eigenfunctions, which are plotted in Fig. 1, show that $y_i(z)$ has no zero in $z > 0$

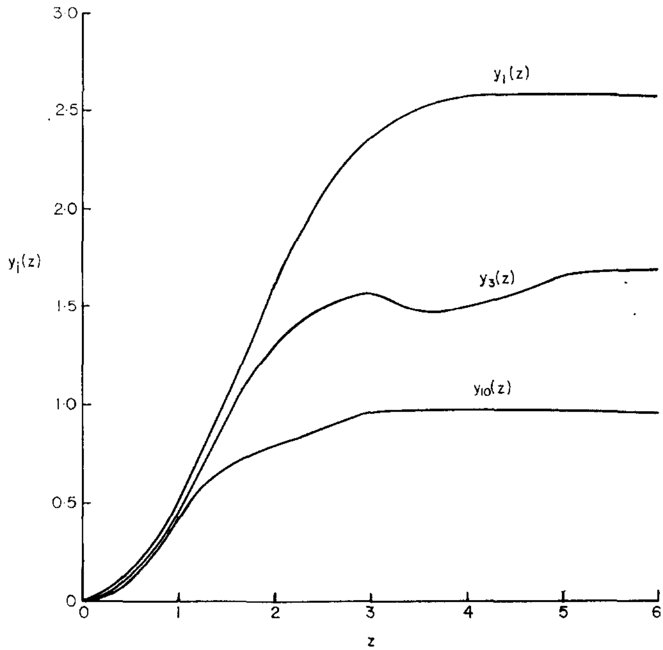

FIG. 1. Blasius eigenfunction $y_i(z)$ ($i = 1, 3, 10$) normalized so that $y''(0) = 1$.

so that for switching procedure 1, $\eta(z) \neq 0$ for $z \geqslant \bar{z}$ and no G singularities are encountered. In procedure 2, $\eta(z) = P(e_1 y(z) + e_2 y'(z) + y''(z))$ and this is dominated by the first term which has no zeros in $z \geqslant \bar{z}$.

The switching method described appears to have advantages over the contour integration method, both in terms of computing time and in its ability to recover the eigenfunction.

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